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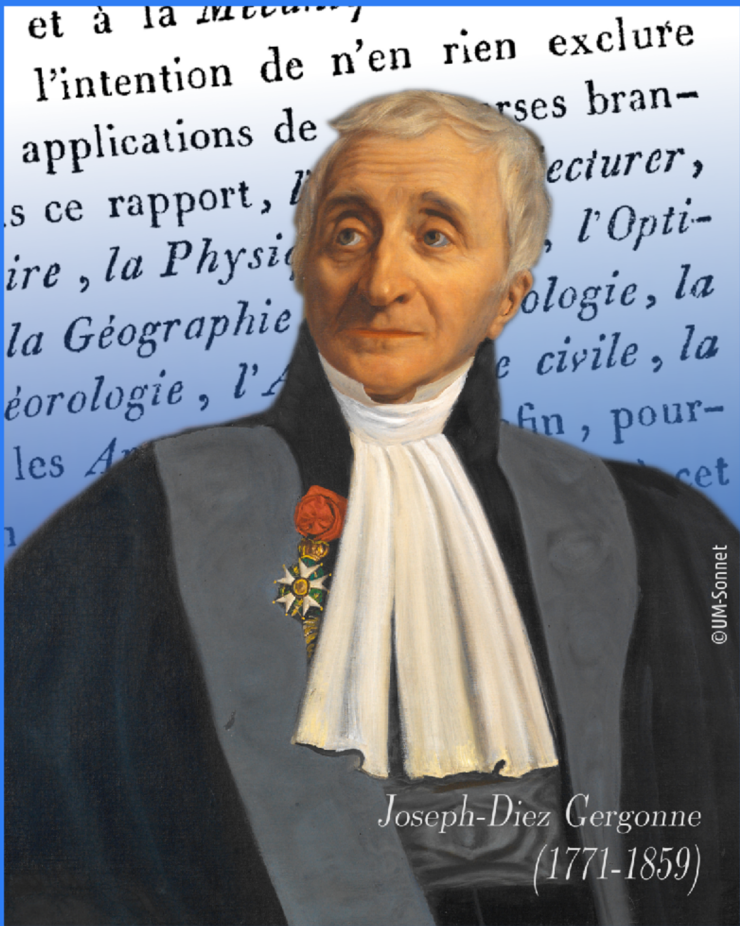

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TOWARDS A CULTURALLY MEANINGFUL HISTORY OF CONCEPTS AND THE ORGANIZATION OF MATHEMATICS TEACHING ACTIVITY

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ABSTRACT

The research aims to investigate possible implications that the relationship between the phylogenesis and the ontogenesis in the organization of the teaching of mathematics may have in teacher education. We seek to establish links between a pedagogical approach to mathematical concepts, the history of the concept, and its cultural signification, basing the relationship between human activity, social practice, and the history of concepts from a historical-cultural perspective. We suggest that the sense of a problem can achieve a new dimension by involving elements from the history of mathematics – as a support for the development of potentially triggering situations of learning, the concept of problem-solving classroom activity, and the symbolic systems of cultural signification. We argue that the history of mathematics allows the recognition of social practices related to the historical and cultural production of concepts. It also allows teachers to understand the limits of mathematical problems that can be formulated and the necessary mediation, in order to help students become aware of theoretical ways of thinking mathematically.

1 Introduction

In this research, we investigate possible implications that the relationship between the phylogenesis and the ontogenesis in the organization of the teaching of mathematics may have in teacher education. We seek to establish links between a pedagogical approach to mathematical concepts, the history of the concept, and its cultural signification. In order to do so, we draw on historical-cultural perspective (Vygotsky, 2002; Leontiev, 1983; Kopnin, 1978) and of the Cultural Theory of Objectification (Radford, 1997, 2006, 2013, 2014).

In this theoretical context, we understand that the learning process of teachers involves the encounter and grasping of concepts, the practices in which these concepts are subsumed, the values that the concepts convey, and the ways of acting and reflecting that encompass and endow with meaning the target concepts. The encounter and grasping of such concepts and their theoretical constellations produced historically and socially is what we term objectification. According to Vygotsky, the encounter of historical concepts takes place in human activity involving signs and tools, in a dialectical movement between inter- and intrapsychic processes (Vygotsky, 2002), and results in the production of sense that relates to a change of motive in the activity developed by the individuals, based on a certain need (Leontiev, 1983).

The concept of need, understood from a dialectic perspective, goes beyond the immediate relationship between individual, need, and objective. Need, from a dialectic perspective bears an ontological meaning that Fraser (1998), drawing on the works of Hegel and Marx, connects to ethical, social, and aesthetical dimensions. Human beings here are seeing as beings of need. It is in the ontological constitution of the individual to find the bases of her existence beyond herself: in nature, in society, and in others. The distinction between natural needs and socially created ones indicates the change in the way every person satisfies their needs (Fraser, 1998, p.125). The concept of human need in Marx relates, indeed, to the realization of the human essence mediated by consciousness (Fraser, 1998, p.143).

The relationship between the learning of teachers and the concomitant process of consciousness —i.e., the process of becoming conscious of cultural meanings (mathematical and others)— implies the transformation of sense and thus the transformation of the needs in the individual's teaching activity.

2 Human Activity, Social Practice, and History of Mathematics

From a historical and cultural perspective (e.g., Vygotsky, 1989, 2002; Leontiev, 1983, 2001; Moura, 2007; Radford, 1997, 2006, 2013, 2014), mathematical concepts are understood as human productions that aim at meeting the needs of individuals at a certain historical time and place.

One example of the relationship between the production of mathematical knowledge and their corresponding human activities and social practices can be found in the study developed by Høyrup (1994) on the history of measure, number, and weight in the cultures of Mesopotamia and Greece. By overcoming a platonic understanding of mathematics, Høyrup shows how cultural institutions mediate the influence of general sociocultural forces on individuals at the same time those individuals also contribute to modelling the interaction with the sociocultural forces. In order to make explicit such mediation (even if the latter is not recognized as such, that is, in its historical-dialectic context), Høyrup identifies the sense of the work of the scribes in their social and historical context. More specifically, Høyrup shows that, in spite of the demand concerning the immediate needs of everyday life, the scribes' motivation to solve problems went through a social *recognition* and the *professional identity* of that activity, so that "scribal practice transposed from the region of practical necessity into that of virtuosity" (Høyrup, 1994, p. 66), which was only possible in a certain society that valued and encouraged it – and, therefore, in an imbricated way or subsumed to social systems of the production of sense.

Another aspect explored by Høyrup is the constitution of mathematics as an entity and a field of knowledge as the "point where preexistent and previously independent mathematical practices are coordinated through a minimum of at least intuitively grasped understanding of formal relations" (Høyrup, 1994, p.67-68).

Such understanding of the relationship between social practices and abstract mathematical knowledge has also been undertaken by Kopnin (1978) on the historical and logical aspects of concepts. The concept of history, as Kopnin (1978) adopts it, as it differentiates from a positivist view on history, meets the concept proposed by the historical-cultural perspective. According to Marx and Engels (1976),

> The first historical act is thus the production of the means to satisfy these needs, the production of material life itself. And indeed this is an historical act, a fundamental condition of all history, which today, as thousands of years ago, must daily and hourly be fulfilled merely in order to sustain human life. […] The second point is that the satisfaction of the first need (the action of satisfying, and the instrument of satisfaction which has been acquired) leads to new needs; and this production of new needs is the first historical act. (Marx and Engels, 1978, p.48)

The concept of history is understood as an ontological category —which constitutes the human— directly connected to the way individuals produce their life and their existence through the production of new needs that overcome the natural needs. Those new needs, intrinsically human, are social, cultural, and historical.

Immersed in this dialectic comprehension of history, Kopnin (1978) contends that the historical movement of the production of concepts, particularly the mathematical ones, as it was developed by humans in sensuous and material activity, constitutes *the logical aspect of the concep*t.

Within this context, the production of mathematical ideas is understood in unity with its signification manifested in social practices in a culturally specific environment. The further production and refinement of mathematical ideas (e.g., the concept of number or the concept of geometric figure) constitute their phylogenesis, that is their historical development *in the dialectic sense of history* mentioned above. Their ontogenesis is the development of these ideas in the course of the individuals' life. But ontogenesis is not the mere repetition of the historical path of the concepts (Furinghetti and Radford, 2008; Radford and Puig, 2007). It is only, as Vygotsky (1989) noted, in the organic realm that such a repetition may take place. In organic development, phylogeny is repeated in ontogeny. In cultural development

> there is a real interaction between phylogeny and ontogeny: man [sic] is not necessary as a biotype: for the human fetus or embryo to develop in the mother's uterus, it is not necessary for it to interact with a mature biotype. In cultural development, this interaction is the principal driving force of all development (adult and child arithmetic, speech, etc.).

It is hence in the unity between phylogenesis and ontogenesis that we find the driving force of cultural development; the former "revives" the latter in the logical sense of the concept, as proposed by Kopnin (1978). From this perspective, the concept is understood in the unity between logic and its (previous, but also new and creative) use in human activity. That is to say, the concept is the unit between phylogenesis and ontogenesis.

Drawing on these key ideas of dialectic materialism, we have already stressed the importance of the interaction between sociocultural history and the ontogenetic development of culture and its individuals. As pointed out in a previous article "to understand conceptual developments we need to place the cognizer and the whole mathematical activity under study within his or her cultural conception of mathematics and of science in general". (Radford, 1997, p.28). From this viewpoint, it is necessary to recognize the importance of studying concepts in their process of production, along with the cultural significations intrinsic to the culture in which they are inserted, since, ontogenetically, human thinking is subsumed into a cultural reality (Radford, 1997, 2006, 2013, 2014).

The concept of objectification (Radford, 2002) is an attempt to try to understanding knowledge as a cultural objective entity and its relation to individuals as they encounter such a knowledge and try to grasp and to make sense of it.

> Objectification is precisely this social process of progressively becoming aware of the Homeric *eidos*, that is, of something in front of us—a figure, a form—something whose generality we gradually take note of and at the same time endow with meaning. It is this act of noticing that unveils itself through counting and signalling gestures. It is the noticing of something that reveals itself in the emerging intention projected onto the sign or in the kinaesthetic movement which mediates the artefact in the course of practical sensory activity, something liable to become a reproducible action whose meaning points toward this fixed eidetic pattern of actions incrusted in the culture which is the object itself. (Radford, 2007, p. 1791)

3 Implications for Teaching and Learning Activity

The dialectic relationship between phylogenesis and ontogenesis, as it is manifested in the relationship between human activity and the socially and historically constituted mathematical knowledge, indicates the potential of the history of mathematics as a support for the teaching organization that aims at developing the theoretical thinking of students —within everyday life problems, but also *beyond* the focus on everyday life situations. Thus, we propose that the sense of mathematical school problems take a dimension that associates elements from the history of mathematics (its historicity in the dialectic understanding of the term), the concept of classroom activity as that which ensures the dialectic unity of phylogenesis and ontogenesis, and what we term *symbolic systems of cultural significations* (Figure 1). Symbolic systems of cultural significations refer to a supra-symbolic dynamic structure where we find cultural conceptions about mathematical objects, their nature, the social standards of meaning production, the manner in which mathematical investigations are supposed to occur, etc. Symbolic systems of cultural significations organize, at a symbolic level, classroom teaching and learning activity, in particular through the modes of knowledge production and the forms of human collaboration that are nurtured in the classroom.

> [...] is through social practice that [men] produce their ideas (mathematical or otherwise), it is clear that social practice does not operate autonomously by itself: the social practice is steeped in symbolic systems which organize it in different ways. These symbolic systems we call semiotic systems of cultural significance. (Radford, 2014, p.10).

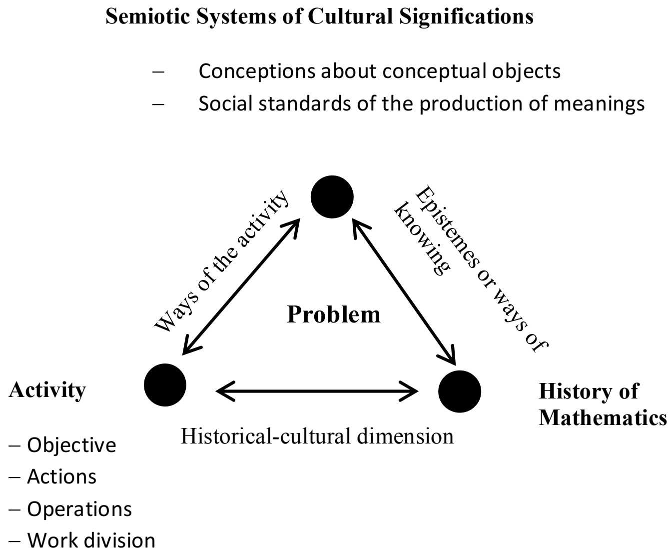

Figure 1. Dimensions of the Problem: History of Mathematics, Activity, and Semiotic Systems of Cultural Significations (adapted from Radford, 2006, p. 109).

Let us turn to an example reported in Janßen and Radford (2015). The example deals with linear equations in the classroom and the manner in which two teachers position themselves vis-à-vis mathematical knowledge and students. Two episodes are discussed. In the first episode the first teacher draws on a mode of knowledge production that can be characterized as *subjectivist*. That is to say, the students are supposed to produce their own knowledge by engaging with the equation that the teacher has chosen for them. The equation is presented in a kind of abstract form, through concrete materials: boxes and matches (see Figure 2). There were 5 boxes and 4 matches on the left side of the equation and 2 boxes and 19 matches on the right side. In symbolic notations, the equation would be translated as $5x + 4 = 2x + 19$.

Figure 2. The teacher (in the middle) discusses with two students an equation expressed through matches an boxes (From Janßen and Radford (2015).

The recourse to concrete material is intended as a means to simplify the epistemological density of the target knowledge. The rules of simplification of the equation, that is the rules of al-gabr and al-muquabala of Al-Khwarizmi (for a discussion of these rules, see Radford, 1993), should appear in their simplicity through the conceptual transparency of the matches and the boxes to the Grade 8 (13–14-year-old) students. As the classroom episode reveals, this is not the case. Yet, the mature biotype (to use Vygotsky's term), that is to say the teacher, is constrained by the very forms of knowledge production she draws on and that lead her to refrain herself from fully interacting with the students. The teacher undergoes a painful process in the course of which she suggests some guilty hints. She manages to utter:

> Well what can one change here for example, so that it stays (*briefly holds both her hands above the two tables*) the same. (*multiply taps the tables with all of her fingers—see Fig. 2, right image*) that must always stay the same that is very important.

The teacher remains imprisoned within the confines of some "gestures, (un)allowable hints, and the unsayable [mathematical] matter" (i.e., that which would be *improper* to mention by the teacher) (Janßen and Radford, 2015). The unit of phylogenesis and ontogenesis is not achieved. The dialectic interaction between phylogenesis and ontogenesis does not happen. The production of the concept was not possible. It took indeed a daring utterance by the teacher to move things a bit forward: against her visible beliefs, talking to the students, she uttered "take away something."

In the second episode reported in the study, the second teacher resorts also to concrete material. But this time, the problem is formulated as a story and the teacher fully interacts with the students, Here is the story problem:

> Sylvain and Chantal have some hockey cards. Chantal has 3 cards and Sylvain has 2 cards. Her mother puts some cards in three envelopes making sure to put the same number of hockey cards in each envelope. She gives 1 envelope to Chantal and 2 to Sylvain. Now, both children have the same amount of hockey cards. How many hockey cards are in an envelope?

Figure 3. The story problem is expressed through envelopes and hockey cards.

The story problem is translated by the teacher in front of the Grade 2 class (7–8-year-old students) and expressed as an equation made up of hockey cards and envelopes (see Figure 3).

In alphanumeric symbols, the equation would be 2x + 2 = x + 3 (a card with the equals sign on it divides the "two sides" of the equation).

The teacher asks for ideas and engages with the ideas that the students offer. She follows the still not fully linguistically articulated actions of Cheb and Cheb's pointing gestures, by moving the concrete envelopes and cards on the blackboard. The reported dialogue (Janßen and Radford, 2015) goes as follows:

1. Teacher: I'll go with the isolating strategy, Ok? Cheb? (*see Fig. 2*)
2. Cheb: Umm… you remove one of Sylvain's envelopes and one of (*the teacher has already put the hand on the envelope, yet she stops to wait for the next part of C's utterance, turning her head towards C)* Chantal's envelopes
3. T: Is it important to remove the same thing from each side of the equal [sign]? (*she makes a two-hand gesture around the equal sign moving the hands to the bottom of the blackboard, where envelopes and cards have been moved, to indicate that removing action is happening in both sides of the equality*)
4. C: Yes. And you can remove the other envelope… Oh non! One of Sylvain's cards and one card from Chantal's *(the teacher removes one card from Chantal's, see Fig. 3, left image*).
5. T: Aw! Again, one envelope, we remove one envelope (*see Fig. 3, centre image, where the teacher points to the removed envelopes*), one card, [and] one card (*see Fig. 3, right image, where the teacher touches the two removed cards*) …
6. C: You remove one of Sylvain's cards and you remove one of Chantal's cards (*the teacher moves the cards towards the top of the blackboard*)
7. T: We remove another card of Chantal's cards. Then, that gives us…
8. C: The answer!

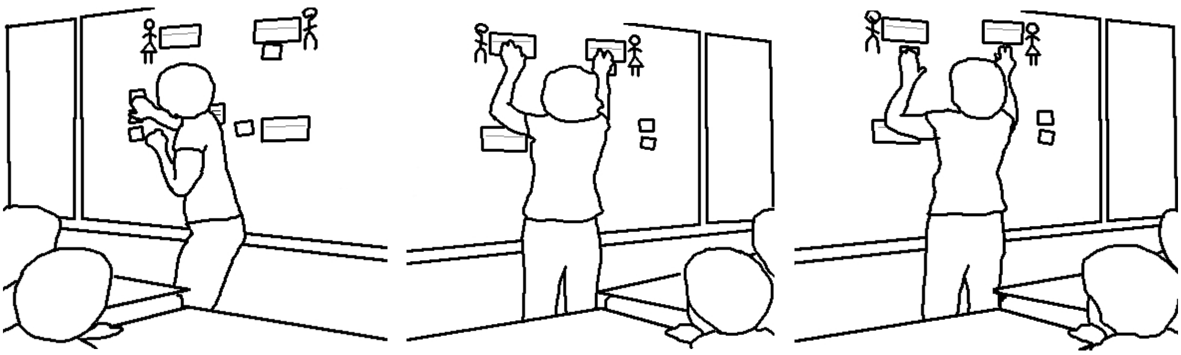

Figure 3. The teacher follows the still not fully linguistically articulated actions of Cheb.

The concrete material is not enough to ensure the encounter of the students with a historically constituted algebraic knowledge. The epistemological density of algebraic knowledge cannot be made transparent by the use of artefacts. A meaningful and challenging situation for the students has to be envisioned. Furthermore, the full participation of the teacher is required. The mature biotype has to participate with the students in order to bring to consciousness the intricate cultural mathematical meanings that underpin algebra and to ensure the unit of phylogenesis and ontogenesis.

The teachers in the short episodes discussed above draw on different semiotic systems of cultural significations. The first teacher draws on learning as an individual and subjective endeavour. The second teacher draws on learning as a social and collective endeavour. They promote different modes of classroom knowledge production and different forms of human cooperation. Needs appear differently. In the first case, need is subjective. In the second case, need is collective. Need is the collective phenomenon driven by the desire to get the problem solved *together*.

4 Concluding Remarks

In this article we have suggested that a meaningfully cultural history of mathematical concepts in mathematics education includes a critical stance towards history. History is not the mere succession of events (Radford, 2016). A meaningfully cultural history of mathematical concepts in mathematics education also includes the recognition of the importance of taking into account the production of mathematical ideas in unity with its signification manifested in social practices in a culturally specific environment—both at the phylogenetic and ontogenetic levels. Within this context, studying the history of mathematics should allow the recognition of the social practices related to the historical and cultural production of concepts as well as the educator's recognition of the limits and the qualitative changes of those practices—which can indicate a theoretical thinking about the practice, without which the production of the concept would not be possible. In the classroom examples presented in this paper, studying history for educational purposes could involve a discussion of the distinction between arithmetic and algebraic methods to solve linear equations. Such a discussion could involve historical problems discussed with teachers to enhance their content knowledge. Knowing the history of mathematics "gives us an idea of the epistemological density of knowledge" and allows us to understand that, for every kind of knowledge, "there is always a possibility already built to think about it", which does not mean repeating it (Radford, in Moretti, Panossian, and Moura, 2015, p. 254).

Such knowledge allows the elaboration of situations that trigger learning and potentially move the students towards a collective need for the concept, as they demand a theoretical thinking on the practice and the recognition of certain historically and culturally signified ways of knowing. Such need is not necessarily related to real historical problems and can emerge from different types of problem situations, such as "a game, a contextualized problem, or even a problem of logical compatibility within mathematics itself" (Moretti and Moura, 2011, p. 443). Needs, as they arise in the classroom, may not be directly related to the real historical problems. Yet, they are deeply entangled with the desires that motivate activity. As Leont'ev noted, "Behind the object [of activity], there always stands a need or a desire, to which [the activity] always answers" (Leont'ev, 1974, p. 22).

Another aspect related to the contribution of the history of mathematics for the teaching organization, from a historical-cultural perspective, concerns the teacher's recognition of a historical and epistemological perspective of knowledge, without which

> [...] we risk not understanding the difficulties that many students may undergo as they meet those condensed ways of reflecting and acting, and we also miss chances to generate sophisticated designs for the activities we wish to bring to the classroom. (Interview with Luis Radford in Moretti, Panossian, and Moura, 2015, p.254)

Finally, focusing on the training of the mathematics educator, we understand that there is no formulation of mathematical problems that can bring out a certain concept or some knowledge *by itself*. The proposition of problems based on the history of mathematics, as we see it, can only be a learning trigger through a *joint work* with the teacher. In this sense, the history of mathematics is clarifying, since it allows the teacher to understand the boundaries of the mathematical problems that can be formulated, as well as the mediation necessary for the student to become creatively aware of theoretical ways of thinking mathematically.

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REFERENCES

Fraser, I. (1998). *Hegel and Marx: The concept of need*. Edinburgh University Press.

Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. English (Ed.), *Handbook of international research in mathematics education (2nd edition)* (pp. 626 - 655). New York: Taylor and Francis.

Høyrup, J. (1994). *In measure, number, and weight: Studies in mathematics and culture*. SUNY Press.

Janßen, T., & Radford, L. (2015). Solving equations: Gestures, (un)allowable hints, and the unsayable matter. In *Proceedings of the ninth congress of the european society for research in mathematics education (CERME)* (pp. 419-425). Prague: Charles University.

Kopnin, P. V. (1978). *A dialética como lógica e teoria do conhecimento* [Dialectic as logic and theory of knowledge]. Rio de Janeiro: Editora Civilização Brasileira.

Leont'ev, A. N. (1974). The problem of activity in psychology. *Soviet Psychology*, *13*(2), 4-33.

Leontiev, A. N. (1983). *Actividad, Conciencia, Personalidad*. Ciudad de La Habana: Editorial Pueblo Y Educación.

Leontiev, A. N. (2009). The Theory of the Development of the Child's. In A. N. Leontev, *The Development of Mind: Selected Works*. Marxists Internet Archive.

Marx, K., & Engels, F. (1978). *The German Ideology*. New York: International Publishers.

Moretti, V. D., & Moura, M. O. de (2011). Professores de Matemática em Atividade de Ensino: Contribuições da Perspectiva Histórico-Cultural para a Formação Docente [The teaching activity of Mathematics teachers: contributions of the cultural-historical perspective for teachers training]. *Ciência & Educação*, *17*(2), 435-450.

Moretti, V. D., Panossian, M. L., & Moura, M. O. de (2015). Educação, educação matemática e teoria cultural da objetivação: Uma conversa com Luis Radford [Education, mathematics education and the cultural theory of objectification: A conversation with Luis Radford]. *Educação e Pesquisa*, *41*(1), 243-260.

Moura, M. O. de (2007). Matemática na infância [Mathematics in childhood]. In M. R. Migueis, & M. G. Azevedo (Eds.), *Educação Matemática na infância: Abordagens e desafios* [Mathematics education in childhood: Approaches and challenges] (pp. 39-64). Vila Nova de Gaia: Gailivro.

Piaget, J., & Garcia, R. (1987). *Psicogénese e história das ciências*. Lisboa: Dom Quixote.

Radford, L. (1993). L'émergence et le développement conceptuel de l'algèbre [The emergence and conceptual development of algebra]. In E. Barbin (Ed.), *Proceedings of the first european summer university* (pp. 69-83). Montpellier: IREM de Montpellier.

Radford, L. (1997). On Psychology, Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics. *Forthe Learning of Mathematics*, *17*(1), 26-33.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14-23.

Radford, L. (2006). Elementos de una teoría cultural de la objetivación. *Revista Latinoamericana de Investigación en Matemática Educativa*, *Special Issue on Semiotics, Culture and Mathematical Thinking*, 103-129.

Radford, L. (2007). Towards a cultural theory of learning. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the fifth congress of the european society for research in mathematics education (CERME – 5)* (pp. 1782-1797). Cyprus: Larnaca.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7-44.

Radford, L. (2014). Cultura e historia: Dos conceptops dificiles y controversiales en aproximaciones contemporaneas en la educación matemática [Culture and history: Two difficult and controversial concepts in current approaches to mathematics education]. In I. A. Mendes, & C. A. Farias (Eds.), *Cultura, Práticas Sociais e Educação Matemática* (pp. 49-68). São Paulo: Livraria da Física.

Radford, L. (2016). Epistemology as a research category in mathematics teaching and learnig. In B. Hodgson, A. Kuzniak, & J. Lagrange (Eds.), *The didactics of mathematics: Approaches and issues* (pp. 31-41). Switzerland: Springer.

Radford, L., & Puig, L. (2007). Syntax and meaning as sensuous, visual, historical forms of algebraic thinking. *Educational Studies in Mathematics*, *66*, 145-164.

Vygotsky, L. S. (1989) Concrete human psychology. *Soviet psychology*, *27*(2), 53-77.

Vygotsky, L. S. (2002). *A formação social da mente* [The social formation of mind]. 6 ed. São Paulo: Martins Fontes.